

The QCD improved electroweak parameter ρ

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In the present paper, we make a detailed analysis for the QCD corrections to the electroweak ρ parameter by applying the principle of maximum conformality (PMC). As a comparison, we show that under the conventional scale setting, we have $\Delta\rho|_{N^3LO} = (8.257^{+0.045}_{-0.012}) \times 10^{-3}$ by varying the scale $\mu_r \in [M_t/2, 2M_t]$. By defining a ratio, $\Delta R = \Delta\rho/3X_t - 1$, which shows the relative importance of the QCD corrections, it is found that its scale error is $\sim \pm 9\%$ at the two-loop level, which changes to $\sim \pm 4\%$ at the three-loop level and $\sim \pm 2.5\%$ at the four-loop level, respectively. These facts well explain why the conventional scale uncertainty constitutes an important error for estimating the ρ parameter. On the other hand, by applying the PMC scale setting, the four-loop estimation $\Delta\rho|_{N^3LO}$ shall be almost fixed to 8.228×10^{-3} , which indicates that the conventional scale error has been eliminated. We observe the pQCD convergence for the ρ parameter has also been greatly improved due to the elimination of the divergent renormalon terms. As applications of the present QCD improved ρ parameter, we show the shifts of the W -boson mass and the effective leptonic weak-mixing angle due to $\Delta\rho$ can be reduced to $\delta M_W|_{N^3LO} = 0.7$ MeV and $\delta \sin^2 \theta_{\text{eff}}|_{N^3LO} = -0.4 \times 10^{-5}$.

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I. INTRODUCTION

The ρ parameter, being defined as the ratio between the strengths for the charged and neutral currents [1], plays an important role for the electroweak physics. A precise determination of ρ can further improve the accuracy of the electroweak precision observables (EWPOs), such as it provides strong indirect constraints for the top quark M_t [2–4] and the Higgs mass M_H [5–7]. Thus, it is helpful to derive a more accurate ρ parameter for precision test of standard model (SM) and for finding new physics beyond SM.

The ρ parameter can be schematically written as

$$\rho = 1 + \Delta\rho. \quad (1)$$

At the Born level, $\rho|_{\text{Born}} = 1$, and the shift of ρ caused by loop-corrections can be defined as, $\Delta\rho = \Pi_Z(0)/M_Z^2 - \Pi_W(0)/M_W^2$, where $\Pi_Z(0)$ and $\Pi_W(0)$ are transversal parts of W -boson and Z^0 -boson self-energies at the zero momentum transfer. At present, the one-loop QCD corrections [1], the two-loop QCD corrections [8–10], the three-loop QCD corrections [11–14], and the four-loop QCD corrections [15–18] to the ρ parameter have been done in the literature. All those improvements on loop calculations provide us great chances for deriving more accurate QCD estimation for ρ .

Under the conventional scale setting, there is renormalization scheme and scale ambiguity for a fixed-order pQCD correction. That is, conventionally, one always takes $\mu_r = Q$ (Q being the typical momentum flow of the process) as its central value, and then varies the scale within a certain region, e.g. $\mu_r \in [Q/2, 2Q]$, to ascertain the scale uncertainty. More specifically, we shall show

that the conventional scale uncertainty for the ρ parameter is still large even for the four-loop level, thus, it is important to find a reliable way to suppress, or even eliminate, such large scale uncertainty.

The principle of maximum conformality (PMC) [19–27] has been designed for eliminating the renormalization scale ambiguity via a systematic way. By applying the PMC scale setting, all the non-conformal terms in perturbative QCD series are summed into the running coupling, and one obtains a unique, scale-fixed and scheme-independent prediction at any finite order. We shall try to eliminate the renormalization scale ambiguity for $\Delta\rho$ by using the PMC R_δ -scheme [26, 27]. The PMC scales are formed by absorbing the $\{\beta_i\}$ -terms that govern the behavior of the running coupling via the renormalization group equation into the running coupling. Those $\{\beta_i\}$ -terms that are related to the quark mass renormalization and etc. should be kept as a separate during the PMC scale setting. To avoid the confusion of using PMC, one can first transform expressions in terms of $\overline{\text{MS}}$ -quark mass into those of on-shell quark mass [28] and then apply PMC. In the present paper, we shall explain this treatment in detail.

The remaining parts of the paper are organized as follows. In Sec.II, we give our calculation technology for $\Delta\rho$ and show how to deal with it within the framework of PMC. In Sec.III, we present our numerical results for $\Delta\rho$, and also present the application of $\Delta\rho$ for both the shift of the W -boson mass $\delta M_W|_{N^3LO}$ and the shift of the effective leptonic weak-mixing angle $\delta \sin^2 \theta_{\text{eff}}|_{N^3LO}$ up to four-loop QCD corrections. The final section is reserved for a summary.

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II. CALCULATION TECHNOLOGY FOR THE QCD CORRECTIONS TO THE ρ PARAMETER

For the conventional scale setting, the renormalization scale μ_r is fixed to be an initial value μ_r^{init} , which is usually chosen as the typical momentum transfer of the process. While for PMC, the value of μ_r^{init} is arbitrary. Thus, in order to apply PMC properly, we shall first transform the four-loop expression for $\Delta\rho$ derived in Refs.[15–18] into those with full initial scale dependence, in which both the singlet and non-singlet contributions shall be taken into consideration.

$$\Delta\rho = 3X_t \left[1 + c_{1,0}(\mu_r^{\text{init}})a_s(\mu_r^{\text{init}}) + \left(c_{2,0}(\mu_r^{\text{init}}) + c_{2,1}(\mu_r^{\text{init}})n_f \right) a_s^2(\mu_r^{\text{init}}) + \left(c_{3,0}(\mu_r^{\text{init}}) + c_{3,1}(\mu_r^{\text{init}})n_f + c_{3,2}(\mu_r^{\text{init}})n_f^2 \right) a_s^3(\mu_r^{\text{init}}) + \mathcal{O}(a_s^4) \right], \quad (2)$$

where $X_t = (G_F M_t^2)/(8\sqrt{2}\pi^2)$ stands for the one-loop result [1], $a_s(\mu_r^{\text{init}}) = \alpha_s(\mu_r^{\text{init}})/4\pi$ and G_F is the Fermi constant. The coefficients $c_{i,j}(\mu_r^{\text{init}})$ are put in the Appendix. The n_f -series in Eq.(2) can be unambiguously associated with the $\{\beta_i\}$ -terms that rightly govern the running behavior of the coupling constant via the R_δ -scheme [26, 27]. In the R_δ -scheme, an arbitrary constant $-\delta$ is subtracted in addition to the standard subtraction $\ln 4\pi - \gamma_E$ for the $\overline{\text{MS}}$ -scheme. The δ -subtraction defines an infinite set of new $\overline{\text{MS}}$ -like renormalization schemes. The β -function of the coupling constant within any R_δ -scheme is the same as the usual $\overline{\text{MS}}$ one. All R_δ -schemes are connected to each other by a scale-displacement rela-

Then, we transform the $\Delta\rho$ parameter with the $\overline{\text{MS}}$ quark masses into the one with the on-shell quark masses. This transformation is important to separate out the right $\{\beta_i\}$ -terms that govern the behavior of the running coupling. The relation between the $\overline{\text{MS}}$ -quark mass and the on-shell quark mass up to three-loop level can be found in Refs.[29–35]. After doing such transformation, all remaining $\{\beta_i\}$ -terms are rightly pertained to the running coupling and the PMC scales can be readily determined.

More explicitly, we write down $\Delta\rho$ up to order $\mathcal{O}(a_s^4)$ in the following:

tion, e.g. for the schemes with δ_1 and δ_2 , their coupling constants are related by

$$a_s(\mu_{\delta_1}) = a_s(\mu_{\delta_2}) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n a_s(\mu_r)}{(d \ln \mu_r^2)^n} \Big|_{\mu_r=\mu_{\delta_2}} (-\delta)^n,$$

where $\ln \mu_{\delta_1}^2 / \mu_{\delta_2}^2 = -\delta$. At each perturbative order, the running behavior of the coupling constant is controlled by such displacement relation, which inversely determines the $\{\beta_i\}$ -terms that pertain to a specific perturbative order. By collecting up all those $\{\beta_i\}$ -terms for the same order, one can obtain the general pattern of non-conformal $\{\beta_i\}$ -terms at each perturbative order. More specifically, according to the R_δ -scheme, we can rewrite Eq.(2) as

$$\Delta\rho = 3X_t \left[1 + r_{1,0}(\mu_r^{\text{init}})a_s(\mu_r^{\text{init}}) + \left(r_{2,0}(\mu_r^{\text{init}}) + \beta_0 r_{2,1}(\mu_r^{\text{init}}) \right) a_s^2(\mu_r^{\text{init}}) + \left(r_{3,0}(\mu_r^{\text{init}}) + \beta_1 r_{2,1}(\mu_r^{\text{init}}) + 2\beta_0 r_{3,1}(\mu_r^{\text{init}}) + \beta_0^2 r_{3,2}(\mu_r^{\text{init}}) \right) a_s^3(\mu_r^{\text{init}}) + \mathcal{O}(a_s^4) \right], \quad (3)$$

where $\beta_0 = 11 - \frac{2}{3}n_f$, $\beta_1 = 102 - \frac{38}{3}n_f$, and the coefficients $r_{i,j}(\mu_r^{\text{init}})$ can be derived from the coefficients $c_{i,j}(\mu_r^{\text{init}})$ defined in Eq.(2), which are also put in the Appendix. The $r_{i,0}$ with $i=(1,2,3)$ are conformal coefficients, and the $r_{i,j}$ with $1 \leq j \leq i \leq 3$ are non-conformal ones that should be absorbed into the running coupling. After absorbing all those non-conformal terms into the running coupling, we finally obtain the scheme-independent conformal series for $\Delta\rho$, i.e.

$$\Delta\rho = 3X_t \left[1 + r_{1,0}(\mu_r^{\text{init}})a_s(Q_1) + r_{2,0}(\mu_r^{\text{init}})a_s^2(Q_2) \right.$$

$$\left. + r_{3,0}(\mu_r^{\text{init}})a_s^3(Q_3) + \mathcal{O}(a_s^4) \right]. \quad (4)$$

Here Q_i with $i = (1, 2, 3)$ are PMC scales. At each perturbative order, there are new types of $\{\beta_i\}$ -terms, so we should introduce new PMC scales at each perturbative order so as to absorb all the $\{\beta_i\}$ -terms into the running coupling consistently [25]. The PMC scales Q_1 and Q_2 can be written as

$$Q_1 = \mu_r^{\text{init}} \exp \left(\frac{1}{2} \frac{-r_{2,1} + \frac{1}{2} \frac{\partial \beta}{\partial a_s} r_{3,2}}{r_{1,0} - \frac{1}{2} \frac{\partial \beta}{\partial a_s} r_{2,1}} \right), \quad (5)$$

$$Q_2 = \mu_r^{\text{init}} \exp\left(-\frac{1}{2} \frac{r_{3,1}}{r_{2,0}}\right), \quad (6)$$

where $\beta = -a_s^2 \sum_{i=0}^{\infty} \beta_i a_s^i$. There is no higher-order $\{\beta_i\}$ -terms to determine Q_3 , we set its value as μ_r^{init} . This treatment causes residual scale dependence, which, however, can be highly suppressed¹.

III. NUMERICAL RESULTS AND DISCUSSIONS

To do numerical calculation, we take the top-quark pole mass $M_t = 173.3$ GeV [38], which is compatible with the $\overline{\text{MS}}$ mass $\overline{m}_t(\overline{m}_t) = 163.3$ GeV [39]. The W -boson mass $M_W = 80.385$ GeV and the Z^0 -boson mass $M_Z = 91.1876$ GeV [40]. The Fermi constant $G_F = 1.16638 \times 10^{-5} \text{GeV}^{-2}$. To be consistent, as an estimation of $\Delta\rho$ up to certain QCD loop correction, we will use different Λ_{QCD} determined by using world average $\alpha_s(M_Z) = 0.1184$ [40]: we use $\Lambda_{\text{QCD}}^{(n_f=5)} = 0.213$ GeV, and $\Lambda_{\text{QCD}}^{(n_f=6)} = 0.0904$ GeV for three-loop α_s running; $\Lambda_{\text{QCD}}^{(n_f=5)} = 0.231$ GeV and $\Lambda_{\text{QCD}}^{(n_f=6)} = 0.0938$ GeV for two-loop α_s running; $\Lambda_{\text{QCD}}^{(n_f=5)} = 0.0899$ GeV and $\Lambda_{\text{QCD}}^{(n_f=6)} = 0.0437$ GeV for the one-loop α_s running.

As a subtle point, as shown by Eqs.(5,6), the PMC scales themselves are in perturbative series, thus the PMC scales shall be improved to a certain degree for the estimation with more and more QCD loop corrections being included. For example, to determine PMC scale Q_1 for $\Delta\rho$ up to three-loop level, we have only β_0 term to determine its value; while for $\Delta\rho$ up to four-loop level, we have both β_0 and β_1 terms to determine its value. Within the PMC scale Q_1 , the β_1 terms are α_s suppressed in comparison to the leading β_0 terms and such difference shall further be exponentially suppressed, thus, its value changes slightly.

A. The QCD improved electroweak ρ parameter

μ_r^{init}	Conventional			PMC		
	$M_t/2$	M_t	$2M_t$	$M_t/2$	M_t	$2M_t$
ΔR_{NLO}	-0.109	-0.098	-0.091	-0.131	-0.131	-0.131
$\Delta R_{\text{N}^2\text{LO}}$	-0.121	-0.118	-0.112	-0.127	-0.127	-0.127
$\Delta R_{\text{N}^3\text{LO}}$	-0.124	-0.123	-0.118	-0.126	-0.126	-0.126

TABLE I. Initial scale dependence for the ratio ΔR_i under the conventional scale setting ($\mu_r \equiv \mu_r^{\text{init}}$) and the PMC scale setting, where $i=\text{NLO}$, N^2LO and N^3LO stand for the QCD corrections to the $\Delta\rho$ parameter up to two-loop, three-loop, and four-loop levels, respectively.

After the PMC scale setting, we obtain a more steady prediction over the scale changes for $\Delta\rho$. To show this point more clearly, we define a parameter:

$$\Delta R_i = \frac{\Delta\rho|_i}{3X_t} - 1, \quad (7)$$

where $i=\text{NLO}$, N^2LO , N^3LO stand for the QCD corrections to the $\Delta\rho$ parameter up to two-loop, three-loop, and four-loop levels, respectively. The scale dependence for ΔR_i under the conventional scale setting and the PMC scale setting are put in Table I, where three typical initial scales $\mu_r^{\text{init}} = M_t/2$, M_t and $2M_t$ are adopted.

We can see from Table I that under the conventional scale setting, the QCD corrections for ΔR_i shows a strong dependence on the choice of (initial) scale μ_r^{init} . For example, its conventional scale error is $\sim \pm 9\%$ for $\mu_r^{\text{init}} \in [M_t/2, 2M_t]$ at the two-loop level, which changes to $\sim \pm 4\%$ at the three-loop level and $\sim \pm 2.5\%$ at the four-loop level. Thus, the scale uncertainty constitutes a systematic error for $\Delta\rho$. In contrast, after applying the PMC scale setting, the value of ΔR_i is almost unchanged for $\mu_r^{\text{init}} \in [M_t/2, 2M_t]$ even at the two-loop level. In fact, by using the formulas (5), (6), it is found that the PMC scales themselves are almost fixed, i.e. for any μ_r^{init} ,

$$Q_1 \simeq 26.2 \text{GeV} \text{ and } Q_2 \simeq 84.6 \text{GeV}. \quad (8)$$

The conformal coefficients $r_{i,0}$, as shown in the Appendix, are also independent of μ_r^{init} .

We present the dependence of $\Delta\rho$ over μ_r^{init} before and after the PMC scale setting in Figs. 1 and 2, where the solid, dashed and dotted lines stand for QCD corrections up to NLO/two-loop, N^2LO /three-loop and N^3LO /four-loop levels, respectively. Fig.1 shows that as one includes higher-and-higher orders, the scale uncertainty will be decreased accordingly to a certain degree: I) By setting $\mu_r = M_t$, we obtain $\Delta\rho \simeq 8.49 \times 10^{-3}$, 8.30×10^{-3} , and 8.26×10^{-3} at the two-loop, three-loop and four-loop levels, respectively; II) By setting $\mu_r = M_t/2$, we obtain $\Delta\rho \simeq 8.39 \times 10^{-3}$, 8.27×10^{-3} , and 8.24×10^{-3} at the two-loop, three-loop and four-loop levels, respectively. Those results agree with the conventional wisdom that by finishing a higher-order enough calculation, one can finally achieve desirable convergent and scale-invariant estimations. It is often argued that by varying the scale, one can estimate contributions from higher-order terms under the conventional scale setting. However, this procedure can only partly estimate the higher-order contributions, since it only partly exposes the $\{\beta_i\}$ -dependent non-conformal terms, not the entire perturbative series [25]. More explicitly, by varying $\mu_r^{\text{init}} \in [M_t/2, 2M_t]$, Table I shows the central value of $\Delta R_{\text{N}^2\text{LO}}$ is not within the error of ΔR_{NLO}

¹ There is another type of residual scale dependence for the already determined PMC scales, which are smaller and are highly exponentially suppressed [36]. This is the reason why the large- β_0 approximation suggested in the literature provides a good approximation for setting the scale in certain processes [37].

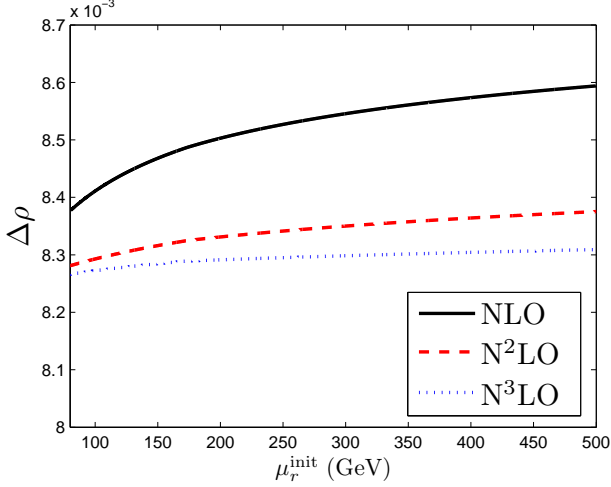


FIG. 1. The $\Delta\rho$ parameter versus the initial renormalization scale μ_r^{init} under the conventional scale setting. The solid, dashed and dotted lines stand for QCD corrections up to NLO/two-loop, $N^2\text{LO}$ /three-loop and $N^3\text{LO}$ /four-loop, respectively.

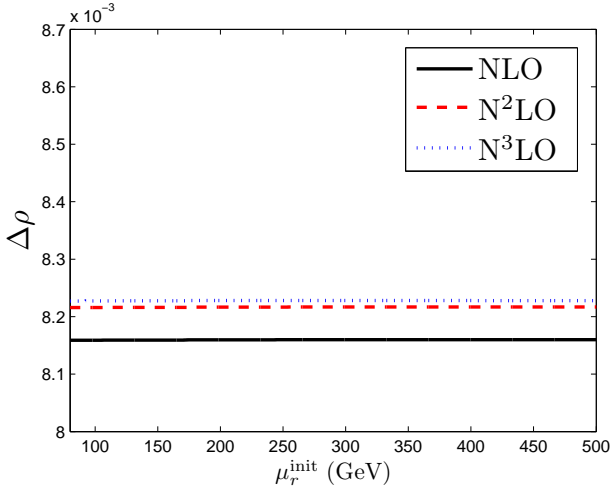


FIG. 2. The $\Delta\rho$ parameter versus the initial renormalization scale μ_r^{init} under the PMC scale setting. The solid, dashed and dotted lines stand for QCD corrections up to NLO/two-loop, $N^2\text{LO}$ /three-loop and $N^3\text{LO}$ /four-loop, respectively.

and the central value of $\Delta R_{N^3\text{LO}}$ is also not within the error of $\Delta R_{N^2\text{LO}}$.

On the other hand, after applying the PMC scale setting, we obtain $\Delta\rho \simeq 8.17 \times 10^{-3}$, 8.22×10^{-3} , and 8.23×10^{-3} up to two-loop, three-loop and four-loop QCD corrections, respectively. Fig.2 shows the $\Delta\rho$ with QCD corrections up to NLO, $N^2\text{LO}$ and $N^3\text{LO}$ are almost flat versus the initial scale μ_r^{init} . It shows that after the PMC scale setting, the value of $\Delta\rho$ shows a faster steady behavior by including higher-and-higher order corrections, which quickly approaches its steady value with more-and-more loop corrections included. One may even estimate

that $\Delta\rho = 8.23 \times 10^{-3}$ could be the final pQCD estimations even by including up to infinite order corrections. To show how the theoretical prediction changes when more and more loop corrections are included, we define a ratio

$$\kappa_i = \left| \frac{\Delta R_i - \Delta R_{i-1}}{\Delta R_{i-1}} \right|,$$

where $i = N^2\text{LO}, N^3\text{LO}$, respectively. This ratio exactly shows how a ('newly') available higher-order correction could be varied from the ('known') lower-order estimation. Under the conventional scale setting, we have

$$\kappa_{N^2\text{LO}} = 11\%, \quad \kappa_{N^3\text{LO}} = 2\% \text{ for } \mu_r^{\text{init}} = M_t/2 \quad (9)$$

$$\kappa_{N^2\text{LO}} = 20\%, \quad \kappa_{N^3\text{LO}} = 4\% \text{ for } \mu_r^{\text{init}} = M_t \quad (10)$$

$$\kappa_{N^2\text{LO}} = 23\%, \quad \kappa_{N^3\text{LO}} = 5\% \text{ for } \mu_r^{\text{init}} = 2M_t \quad (11)$$

While, after the PMC scale setting, we have

$$\kappa_{N^2\text{LO}} \simeq 3\%, \quad \kappa_{N^3\text{LO}} \simeq 0.8\% \text{ for } \mu_r^{\text{init}} \in [M_t, 2M_t]. \quad (12)$$

Moreover, we note that:

- The PMC scale at each perturbative order is determined by absorbing particular $\{\beta_i\}$ -terms into the running coupling. Other than a guess work for the conventional scale setting, the PMC scales and hence the PMC estimations are highly independent of the choice of initial scale μ_r^{init} . Thus, the conventional renormalization scale ambiguity is solved.
- A comparison of Table I indicates that after absorbing the non-conformal terms into the running coupling by PMC, the leaving conformal terms shall provide slight positive contributions from the two-loop level, so $\Delta\rho|_i$ shall be increased when more-and-more loop corrections being included. While, under the convention scale setting, the combination of both the conformal and the non-conformal terms shall always provide negative contributions, so $\Delta\rho|_i$ shall decrease with more-and-more loop corrections being included.
- Another important feature of PMC scale setting is that its final estimation is conformal series and is renormalization scheme independent [25], thus the renormalization scale and scheme dependence under the conventional scale setting are eliminated at the same time.
- For any scale-setting method, we need to finish a full higher-order calculation so as to estimate the magnitude of the conformal terms. The PMC provides a systematic way to estimate the unknown conformal contributions via the extended renormalization group equations [41].

As a byproduct, from a comparison of Table I, we show that if setting $\mu_r \sim M_t/2$ for the conventional scale setting, one can get the same estimation under the PMC

scale setting. Thus, the effective momentum flow for the whole process is $\sim M_t/2$ other than the conventionally suggested M_t , or equivalently, it is $\sim M_t/2$ that can rightly eliminate the large logs and get a more convergent/correct pQCD estimation. There are some other examples also show that the conventional choice of scale is really a guess work. Ref.[28] indicates that the effective momentum flow for $H \rightarrow \gamma\gamma$ decay is $\sim 2M_H$ other than M_H . Ref.[42] argues that after including the first and second order corrections to several deep inelastic sum rules which are due to heavy flavor contributions, the effective scale μ_r for the deep inelastic sum rules should be $\sim 6.5m_Q$ other than m_Q (m_Q being the heavy quark mass). All those indicate that it is clearly artificial to guess a scale Q (we even do not know whether it is the central scale or not) and to study its uncertainty by sim-

ply varying $\mu_r \in [Q/2, 2Q]$, as the conventional scale setting does.

After the PMC scale setting, there is residual scale dependence for the final terms proportional to $a_s^3(Q_3)$, since we have no $\{\beta_i\}$ -terms to determine Q_3 . We can estimate the magnitude of such residual scale dependence following the spirit of PMC scale setting. That is, as suggested in Ref.[43], we rewrite the coupling constant $a_s(Q_3)$ at the four-loop level as follows,

$$a_s(Q_3) = a_s(\mu_r^{\text{init}}) + \beta_0 \ln \left(\frac{(\mu_r^{\text{init}})^2}{Q_3^2} \right) a_s^2(\mu_r^{\text{init}}). \quad (13)$$

Since the log term $\ln(\mu_r^{\text{init}}/Q_3)^2$ can largely compensate the scale changes at the $\mathcal{O}(a_s^3)$ level, and as expected, we obtain a very small residual scale dependence by varying $\mu_r^{\text{init}} \in [M_t/2, 2M_t]$.

	Conventional scale setting				PMC scale setting			
	LO	NLO	N ² LO	N ³ LO	LO	NLO	N ² LO	N ³ LO
$\Delta\rho _i (\times 10^{-3})$	9.411	8.483	8.305	8.257	9.411	8.175	8.217	8.228
$\delta\rho _i (\times 10^{-3})$	-	-0.928	-0.178	-0.048	-	-1.236	0.042	0.011
K_i	-	9.8%	2.1%	0.6%	-	13%	0.5%	0.1%

TABLE II. The parameter $\Delta\rho$, the shift $\delta\rho$, and the K factor before and after the PMC scale setting. $\Delta\rho|_i$ with $i=\text{LO, NLO, N}^2\text{LO}$ and N^3LO denote the QCD corrections up to one-loop, two-loop, three-loop, and four-loop levels, respectively. The $\delta\rho|_i$ and K_i stand for the shift of $\Delta\rho|_i$ and K factor for the two-loop, three-loop or four-loop level, respectively. $\mu_r^{\text{init}} = M_t$.

Finally, we present the QCD correction to $\Delta\rho$ at each perturbative order before and after the PMC scale setting in Table II, where $\Delta\rho|_i$, with $i=\text{LO, NLO, N}^2\text{LO}$, or N^3LO , denote the $\Delta\rho$ with QCD correction up to one-loop, two-loop, three-loop, and four-loop level, respectively. After the PMC scale setting, we obtain a conformal series for $\Delta\rho$, and because of the elimination of renormalons (together with large log-terms), the pQCD convergence can be greatly improved. To show this point more clearly, we define a K factor, whose value at each perturbative order is defined as

$$K_i = \left| \frac{\Delta\rho|_i - \Delta\rho|_{i-1}}{\Delta\rho|_{i-1}} \right| = \left| \frac{\delta\rho|_i}{\Delta\rho|_{i-1}} \right|, \quad (14)$$

where the shift of $\Delta\rho|_i$ is defined as $\delta\rho|_i = (\Delta\rho|_i - \Delta\rho|_{i-1})$, and its values are presented in Table II. The K_i stands for the K factor up to two-loop, three-loop or four-loop level, respectively; that is, $i=\text{NLO, N}^2\text{LO}$, or N^3LO , denotes the QCD correction up to one-loop, two-loop, three-loop, and four-loop level, respectively. The results of K_i are also presented in Table II. The values for K factors decrease much faster after the PMC scale setting, which agree with the above observation that the pQCD convergence can be greatly im-

proved after PMC scale setting. More over, we obtain $\Delta\rho|_{\text{N}^3\text{LO}} = (8.257^{+0.045}_{-0.012}) \times 10^{-3}$ for $\mu_r \in [M_t/2, 2M_t]$ under the conventional scale setting; at the same time, the $\Delta\rho|_{\text{N}^3\text{LO}}$ is almost fixed to be 8.228×10^{-3} after the PMC scale setting.

It is noted that several ways to absorb the $\{\beta_i\}$ -terms into the running coupling have been suggested, such as the PMC-I approach (based on the PMC-BLM correspondence) [20], the PMC R_δ approach [26], and the seBLM approach [44, 45]. A detailed comparison of those approaches can be found in Ref. [46]. At present, we observe the same conclusions as those of Ref. [46]. It is noted that the values of $\Delta\rho$ up to four-loop level agree with each other for those approaches. Because of the choice of different effective $\{\beta_i\}$ -series, there are differences at each perturbative order. More explicitly, if setting the initial scale as $\mu_r^{\text{init}} = M_t$, we obtain two effective scales for those approaches as follows

$$R_\delta \text{ approach : } Q_1 = 26.2\text{GeV}, Q_2 = 84.6\text{GeV},$$

$$\text{PMC - I approach : } Q_1 = 26.3\text{GeV}, Q_2 = 83.5\text{GeV},$$

$$\text{seBLM approach : } Q_1 = 26.1\text{GeV}, Q_2 = 263.7\text{GeV}.$$

The scales $Q_{1,2}$ are almost the same for both the R_δ -approach and the PMC-I approach. Under the seBLM

approach, the effective scale Q_1 is the same as that of the two PMC approaches, but it has a larger Q_2 .

B. Applications of the QCD improved ρ parameter for δM_W and $\delta \sin^2 \theta_{\text{eff}}^{\text{lept}}$

A lot of efforts have been devoted to predict the values of the two important EWPOs as M_W and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ within SM, either theoretically or experimentally [47–50]. At present, the total experimental uncertainties for M_W and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ are $\delta M_W = 15$ MeV [40] and $\delta \sin^2 \theta_{\text{eff}}^{\text{lept}} = 16 \cdot 10^{-5}$ [47]. Recent improvements on Higgs at the LHC [51–54] also allow us to determine the EWPOs with high precision [55–59]. At the future International Linear Collider (ILC), it is estimated that the accurate experimental uncertainties as $\delta M_W = 6$ MeV and $\delta \sin^2 \theta_{\text{eff}}^{\text{lept}} = 13 \cdot 10^{-5}$ can be achieved [48]². On the other hand, theoretically, the dominant shifts of M_W and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ are due to $\Delta\rho$ though the following formulas [16]

$$\begin{aligned} \delta M_W|_i &= \frac{M_W}{2} \frac{c_W^2}{c_W^2 - s_W^2} \delta\rho|_i \\ &= \frac{M_W}{2} \frac{c_W^2}{c_W^2 - s_W^2} (\Delta\rho|_i - \Delta\rho|_{i-1}) \end{aligned} \quad (15)$$

and

$$\begin{aligned} \delta \sin^2 \theta_{\text{eff}}^{\text{lept}}|_i &= -\frac{c_W^2 s_W^2}{c_W^2 - s_W^2} \delta\rho|_i \\ &= -\frac{c_W^2 s_W^2}{c_W^2 - s_W^2} (\Delta\rho|_i - \Delta\rho|_{i-1}), \end{aligned} \quad (16)$$

where $c_W = M_W/M_Z$ and $s_W^2 = 1 - c_W^2$, and sequently, with $i=\text{NLO}$, N^2LO , or N^3LO , respectively.

	Conventional			PMC		
	NLO	N ² LO	N ³ LO	NLO	N ² LO	N ³ LO
$\delta M_W _i$ (MeV)	-52.3	-10.0	-2.7	-69.7	+2.4	+0.7
$\delta \sin^2 \theta_{\text{eff}}^{\text{lept}} _i (\times 10^{-5})$	+29.0	+5.6	+1.5	+38.6	-1.3	-0.4

TABLE III. The shifts δM_W and $\delta \sin^2 \theta_{\text{eff}}^{\text{lept}}$ due to the QCD improved ρ parameter before and after the PMC scale setting, where the symbols NLO, N²LO and N³LO shifts due to the QCD corrections up to two-loop, three-loop, and four-loop levels, respectively. $\mu_r^{\text{init}} = M_t$.

An QCD improved ρ parameter leads to improved estimations on M_W and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$, which shall help us for a more confidential comparison with the experimental results and for searching new physics beyond SM

over those EWPOs. We present the shifts of M_W and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ caused by the QCD corrections to $\Delta\rho$ before and after the PMC scale setting in Table III. At the N³LO level, the shifts are $\delta M_W|_{\text{N}^3\text{LO}} = -2.7$ MeV and $\delta \sin^2 \theta_{\text{eff}}^{\text{lept}}|_{\text{N}^3\text{LO}} = 1.5 \times 10^{-5}$ under the conventional scale setting, whose precision can be greatly improved by about four times after the PMC scale setting due to a more convergent pQCD series, i.e.

$$\delta M_W|_{\text{N}^3\text{LO}} = 0.7 \text{ MeV} \quad (17)$$

and

$$\delta \sin^2 \theta_{\text{eff}}^{\text{lept}}|_{\text{N}^3\text{LO}} = -0.4 \times 10^{-5}. \quad (18)$$

IV. SUMMARY

We have applied PMC to analyze the electroweak ρ parameter up to four-loop QCD corrections. After the PMC scale setting, we obtain a more accurate estimation on $\Delta\rho$ with a better pQCD convergence, and then a better estimation of the two EWPOs as δM_W and $\delta \sin^2 \theta_{\text{eff}}^{\text{lept}}$ can be achieved. More specifically,

- We obtain, $\Delta\rho|_{\text{N}^3\text{LO}} = (8.257_{-0.012}^{+0.045}) \times 10^{-3}$ for $\mu_r \in [M_t/2, 2M_t]$ under the conventional scale setting; while, at the same time, the $\Delta\rho|_{\text{N}^3\text{LO}}$ is almost fixed to be 8.228×10^{-3} after the PMC scale setting. It shows that the conventional scale uncertainty can be eliminated and the pQCD convergence can also be greatly improved by applying the PMC scale setting. Thus, it provides another good example for achieving the optimal renormalization scales of the process via PMC, which has been detailed illustrated in Ref.[23].
- In comparison to the results under the conventional scale setting, after applying the PMC scale setting, we obtain a more steady prediction for the shift of W -boson mass, δM_W , and the shift of the effective leptonic weak-mixing angle, $\delta \sin^2 \theta_{\text{eff}}^{\text{lept}}$. More over, as shown by Eqs.(17,18), the QCD improved shifts $\delta M_W|_{\text{N}^3\text{LO}}$ and $\delta \sin^2 \theta_{\text{eff}}^{\text{lept}}|_{\text{N}^3\text{LO}}$ are well below the precision anticipated even for the future ILC experiment. Thus, we shall have great chances to test SM with high precision.
- To apply the PMC scale setting to higher-order pQCD calculations, it is more convenient to use the expressions of $\Delta\rho$ under the on-shell renormalization scheme, e.g. via using the pole top-quark mass, such that there is no ambiguity in dealing with the n_f series of the process, i.e. only those n_f -terms that rightly determine the running behavior of the running coupling should be absorbed into the running coupling.

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² Through its GigaZ program, $\delta \sin^2 \theta_{\text{eff}}^{\text{lept}}$ can be even improved as $1.3 \cdot 10^{-5}$ [60].

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APPENDIX: THE COEFFICIENTS $c_{i,j}(\mu_r^{\text{init}})$ AND $r_{i,j}(\mu_r^{\text{init}})$

By using the expressions of Refs.[15–18], including both the singlet and non-singlet contributions, the coefficients $c_{i,j}(\mu_r^{\text{init}})$ for $\Delta\rho$ up to four-level are

$$c_{1,0}(\mu_r^{\text{init}}) = -\frac{8}{3} - \frac{8\pi^2}{9}, \quad (19)$$

$$c_{2,0}(\mu_r^{\text{init}}) = -404.981 + 125.836 \ln \frac{M_t^2}{(\mu_r^{\text{init}})^2}, \quad (20)$$

$$c_{2,1}(\mu_r^{\text{init}}) = 28.5794 - 7.62643 \ln \frac{M_t^2}{(\mu_r^{\text{init}})^2}, \quad (21)$$

$$c_{3,0}(\mu_r^{\text{init}}) = -20372.1 + 10076.4 \ln \frac{M_t^2}{(\mu_r^{\text{init}})^2} - 1384.2 \ln^2 \frac{M_t^2}{(\mu_r^{\text{init}})^2}, \quad (22)$$

$$c_{3,1}(\mu_r^{\text{init}}) = 2843.1 - 1313.62 \ln \frac{M_t^2}{(\mu_r^{\text{init}})^2} + 167.782 \ln^2 \frac{M_t^2}{(\mu_r^{\text{init}})^2}, \quad (23)$$

$$c_{3,2}(\mu_r^{\text{init}}) = -73.558 + 38.1059 \ln \frac{M_t^2}{(\mu_r^{\text{init}})^2} - 5.0843 \ln^2 \frac{M_t^2}{(\mu_r^{\text{init}})^2}, \quad (24)$$

where M_t stands for the top-quark pole mass.

The coefficients $r_{i,j}(\mu_r^{\text{init}})$ for $\Delta\rho$ can be divided into two types, one is the conformal type, which includes

$$r_{1,0}(\mu_r^{\text{init}}) = -\frac{8}{3} - \frac{8\pi^2}{9}, \quad (25)$$

$$r_{2,0}(\mu_r^{\text{init}}) = 66.5793, \quad (26)$$

$$r_{3,0}(\mu_r^{\text{init}}) = 1925.76, \quad (27)$$

and the other is the non-conformal type, which includes

$$r_{2,1}(\mu_r^{\text{init}}) = -42.8691 + 11.4396 \ln \frac{M_t^2}{(\mu_r^{\text{init}})^2}, \quad (28)$$

$$r_{3,1}(\mu_r^{\text{init}}) = 95.5017 - 66.5793 \ln \frac{M_t^2}{(\mu_r^{\text{init}})^2}, \quad (29)$$

$$r_{3,2}(\mu_r^{\text{init}}) = -165.506 + 85.7382 \ln \frac{M_t^2}{(\mu_r^{\text{init}})^2} - 11.4396 \ln^2 \frac{M_t^2}{(\mu_r^{\text{init}})^2}. \quad (30)$$

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